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Vector Space Partitions and Designs Part I-Basic Theory

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Abstract. This article, written in two parts, concerns partitions of finite vector spaces of dimension $t+k$ by one subspace of dimension t (the 'focus') and the remaining subspaces of dimension k; a 'focal-spread of type (t,k)'. Focal-spreads of type $(k+1,k)$ also produce $2-(q^{k+1},q,1)$-designs, and various other double and triple-spreads. There are three different methods given to construct focal-spreads, one of which is due to Beutelspacher. In this Part I, we shall also provide a coordinate method for their construction analogous to matrix spread sets for translation planes. In Part II, we shall give a new construction that we term "going up," which also allows a specification of certain subplanes of the focal-spread. Additive focal-spreads are shown to be equivalent to additive partial spreads. Various applications are given relative to additive partial spreads and semifield planes admitting exotic subplanes. Finally, also in Part II, , the developments of focal-spreads may be applied to construct a variety of new subgeometry partitions of projective spaces.

Keywords: vector space partition, designs, focal-spread

MSC 2000 classification: 51E23 (primary), 51A40.

1 Introduction

This article is concerned with partitions of finite vector spaces by vector subspaces not necessarily of the same dimension. Although there are a number of examples of such partitions, it is difficult to develop much of a theory. It is well known that to construct a finite translation plane of order q^t, one needs only to construct a t-spread of a vector space of dimension $2t$ over $GF(q)$, that is, a partition of the vector space by a set of q^t+1, mutually disjoint, t-dimensional vector subspaces. More generally, a translation Sperner space may

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be constructed by relaxing the dimension of the vector space to kt-dimensional over $GF(q)$ and finding a partition of t-dimensional subspaces, this time requiring $(q^{kt}-1)/(q^t-1)$ such subspaces. We shall call such partitions 'Sperner t-spreads' for arbitrary dimension kt (leading to translation Sperner spaces) and 'planar t-spreads' (leading to translation planes), when $k=2$. We call the t-dimensional subspaces 'components' in either situation.

However, more generally, it is possible to consider arbitrary partitions of a finite vector space by mutually disjoint subspaces of various dimensions. One manner of constructing different partitions of a vector space is to start with a Sperner t-spread and further partition or refine a component. For example, one may take a given component and partition into 1-dimensional $GF(q)$-subspaces. In this article, we are more interested in the partitions of vector spaces that cannot arise by such refinements of a given Sperner t-spread. In the 'Handbook of Finite Translation Planes' [6], it is mentioned that there are not very many known partitions of vector spaces, in the sense that the partitions are not refinements of Sperner t-spreads.

The existence of such partitions has been established by Beutelspacher [2], who proves if the dimension of the vector space is n and it is required to find a partition, where the dimensions of the subspaces are $\{t_1, t_2, \ldots, t_k\}$, where $t_i < t_{i+1}$, then if $\gcd\{t_1, t_2, \ldots, t_k\} = d$ and $n > 2t_1([t_k/(d \cdot k)] + t_2 + \cdots + t_k)$, a partition may be constructed with various subspaces of dimension t_i.

One of the first questions that might be considered is whether there are partitions of vector spaces that contain one subspace of dimension t and the remaining subspaces of dimension $k \neq t$, so that there is a partial Sperner k-spread and also what can be said theoretically of such partitions. The short answer is that there many of these partitions, due to a construction of Beutelspacher [2], which arise from t-spreads corresponding to translation planes and which we call 'k-cuts' in this paper and are in vector spaces of dimension $t+k$. In this article, we introduce ideas of translation planes for the analysis of such partitions. In general, we call any partition a 'focal-spread of type (t,k)', where the unique subspace of dimension t is called the 'focus' of the partition. The main thrust of this paper is that by the use of ideas and theory of finite translation planes, there is a way to consider a theory of certain partitions that are not t-spreads.

The nature of our study involves a substantial number of new areas of research and so this paper is written in two parts. In this first paper, Part I, we shall give the basic material. In the second article, Part II, we shall develop two important themes. First we consider a new construction technique for focal-spreads, which we term "going up," and second, we give a complete analysis of how to use focal-spreads for the construction of subgeometry partitions of projective spaces and provide a variety of new examples.

Given a k-cut, of course, there is a corresponding spread for a translation plane that produces it. It is a central and potentially important question to ask if every focal-spread is a k-cut; if it can 'extended' to a spread for a translation plane. In a previous article [8], we have given a very general construction procedure that produces k-spreads in vector spaces of dimension sk over a field isomorphic to $GF(q)$, using a set of translation planes of order q^k. In Part II, we may use similar constructions to construct from a sk-vector space a variety of focal-spreads with focus of dimension $k(s-1) = t$, by the going up process. If such focal spreads then are k-cuts, or rather arise from translation planes of order q^t and their associated t-spreads, then our methods allow us to construct translation subplanes of order q^k in very specific ways. Generally, subplanes of translation planes are not well understood and those that are known tend to be Desarguesian or of the same general type as the superplane. Thus, it seems extremely unlikely that all the focal spreads that arise in the going up process may arise as k-cuts from t-spreads of translation planes, where $t = k(s-1)$ in this context. It is also possible to define a focal-spread simply as a partition of a vector space by one subspace of dimension t' and the remaining subspaces of dimension k, where the vector space has dimension $t+k$, and there $t' \leq t$. We call these partitions focal-spreads of type (t, t', k) if $t' < t$. The going up process allows constructions of this more general type.

We are particularly interested in 'additive focal-spreads', where there is a natural additive structure on the partition. We are able to show that arbitrary additive partial t-spreads are equivalent to additive focal-spreads.

1 Theorem. *Additive focal-spreads of dimension $t+k$ with focus of dimension t over $GF(p)$ are equivalent to additive partial t-spreads of degree p^k.*

For a theoretical consequence of such insights, in a related article [9], the authors show that it is possible to construct additive focal-spreads that are equipped with affine subplanes of order p^t, where the order of the focal-spread is p^n, where t does not divide n. With the exception of a single semifield plane of order 2^5 that admits an affine subplane of order 2^2, all previously known affine subplanes of translation planes never have such a property, this again suggests that there are many focal-spreads that are not k-cuts, since the translation plane providing the k-cut would also admit the type of subplane in question.

However, considering these questions, and some of the ideas presented here, the authors [9] also have shown that there are infinitely many known semifield planes of even order 2^n, for n odd, that admit affine subplanes of order 2^2, where $n = 5k$ or $7k$, for any odd integer k. The semifield planes are the commutative binary semifield planes of Knuth [12] and their generalizations due to Kantor [10]. Furthermore, in the above mentioned article of the authors', additive partial spreads are used to show that either there are many new additive maximal partial spreads or there are great numbers of semifield planes with exotic subplanes yet to be discovered. In Part I, we also give a matrix approach for the analysis of focal-spreads.

So, in this article, we give the basics of focal-spreads and use some theory from translation planes for their study. We also show that the existence of focal-spreads of type $(k+1,k)$ leads to a construction of designs of type $2-(q^{k+1},q,1)$ and to other double-spreads or triple-spreads (partitions where there are subspaces in the partition of either two or three different dimensions, respectively).

Another important use of focal-spreads is in the construction of subgeometry partitions. A 'subgeometry partition' of a projective space is a partition by subgeometries isomorphic to projective spaces. So, in the author's work [5], we use focal-spreads to construct a variety of new and unusual subgeometry partitions.

In the final section of Part II, we review the various sorts of applications of focal-spreads that we have been able to initially determine. There are a number of problems yet to be considered and we also detail several of these.

2 Beutelspacher's Construction

Certainly, the impetus for this paper comes from a construction of Beutelspacher, which is as follows:

Let V_{t+k} be a vector space of dimension $t+k$ over $GF(q)$ for $t>k$ and let L be a subspace of dimension t. Let V_{2t} be a vector space of dimension $2t$ containing V_{t+k} ($t>k$ required here) and let S_t be a t-spread containing L. There are always at least Desarguesian t-spreads with this property. Let M_t be a component of the spread S_t not equal L. Then $M_t \cap V_{t+k}$ is a subspace of V_{t+k} of dimension at least k. But, since M_t is disjoint from L, the dimension is precisely k and we then obtain a focal-spread with focus L. This construction may be found in Beutelspacher [2], Lemma 2, page 205. One of the referees suggested calling the corresponding focal-spread a 'k-cut of a t-spread' and we shall adopt this terminology and use the notation $F = S_t \backslash V_{t+k}$ for the focal-spread F.

The formal definition is

2 Definition. A partition of a finite-dimensional vector space of dimension $t+k$ by a partial Sperner k-spread and a subspace of dimension $t \neq k$, shall be called a 'focal-spread of type (t,k)'. The unique subspace of dimension t of the partition shall be called the 'focus' of the focal-spread.

We define a 'planar extension' of a focal-spread as a t-spread such that the focal-spread is of type (t,k) and arises from the t-spread as a k-cut.

The 'kernel' of a focal-spread of dimension $t+k$, over $GF(q)$, with focus of dimension t, shall be defined as the endomorphism ring of the vector space that leaves the focus and each k-component invariant.

3 Remark. For finite translation planes, the kernel is a field. One of the referee's pointed out that the same result of André (see Lüneburg [13] (1.6), p. 3) shows that the kernel of any partition of a finite vector space is also a field. For focal-spreads, the kernel will also act semi-regularly on the non-zero vectors of each component.

2.1 Focal-Spreads: Coordinate-Free Approach

Let $V = V_t \oplus V_k$ be a vector space of rank $t+k$ expressed as a direct sum of subspaces V_t and V_k, having dimensions t and k respectively. Then a *(t,k)-spread set on V, based on axes* (V_k, V_t), is a collection $\mathcal{S}$ of linear maps from V_k to V_t, such that (1) $0 \in \mathcal{S}$; (2) the nonzero maps in $\mathcal{S}$ are injective; (3) the difference between any two members of $\mathcal{S}$ is injective or zero; (4) $\mathcal{S}$ is transitive in the sense that for any pair of non-zero vectors $x \in V_k, y \in V_t$ there is a unique $S \in \mathcal{S}$ such that $xS = y$. The following remark and proposition follow using the same correspondence between t-spreads and spread-sets.

4 Remark. Every (t,k)-spread set $\mathcal{S}$ as above yields a focal-spread of type (t,k), with component set $\{M_S : S \in \mathcal{S}\} \cup \{V_t\}$, where $M_S := \{(x, xS) : x \in V_k\}$.

5 Proposition. *Every focal-spread of type (t,k) may be coordinatized by a (t,k)-spread set, and any focal-spread of type (t,k) arises by coordinatization by a (t,k)-spread set, which is uniquely determined by the focus and any other component chosen as 'basis'.*

As for planar spreads, it is usually helpful to express these results in terms of matrices. This material follows so closely to analogous matrix coordinatization of translation planes that we leave all of the straight forward proofs to the reader. The reader is directed to the standard texts on translation planes Lüneburg [13], Hughes and Piper [3] or the authors' texts ([1] or [6]) for these concepts.

Let B be a focal-spread of dimension $t+k$ over $GF(q)$ with focus L of dimension t. Fix any k-component M. We may choose a basis so that the vectors have the form $(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_t)$. Let $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_t)$, where the focus L has equation $x = 0 = (0,0,0,\ldots,0)$ (k-zeros) and the fixed k-component M has equation $y = 0 = (0,0,0,\ldots,0)$ (t-zeros). We note that $q^{t+k} - q^t = q^t(q^k - 1)$, which implies that there are exactly q^t k-subspaces in the focal-spread. Consider any k-component N distinct from $y = 0$. There are k basis vectors over $GF(q)$, which we represent as follows: $y = xZ_{k,t}$, where $Z_{k,t}$ is a $k \times t$ matrix over $GF(q)$, whose k rows are a basis for the k-component. Hence, we obtain a set of q^t k-components, which we also represent as follows: Row 1 shall be given by $[u_1, u_2, \ldots, u_t]$, as u_i vary independently over $GF(q)$. Then the rows $2, \ldots, k$ have entries that are linear functions of the u_i.

6 Remark. The $k \times t$ matrices in the focal-spread have rank k and the difference of any two distinct matrices associated with k-components also has rank k.

The following is essentially immediate.

7 Theorem. *Let V_{t+k} be a $t+k$-dimensional vector space over $GF(q)$ and let S be a set of $q^t - 1$ $k \times t$ matrices of rank k such that the difference of any two distinct matrices also has rank k. Then there is an associated focal-spread constructed as*

$$x = 0, y = 0, y = xM; \;\; M \in S,$$

where x is a k-vector and y is a t-vector over $GF(q)$, where the focus is $x = 0$.

In particular, it is possible to choose one k-space to have 1's in the (i,i), position and 0's elsewhere in the $k \times t$ matrix.

Conversely, any focal-spread has such a representation.

8 Remark. Suppose that B is a focal-spread of dimension $t+k$, with focus of dimension $t < k$. Then necessarily the rank of the associated matrices cannot have rank k. Hence, $t > k$ for focal-spreads.

From the matrix spread set point of view, we now actually may define what we mean by **'extension of a focal-spread'**. This would then mean that the $k \times t$ matrices for a focal-spread have been extended to a set of $t \times t$ matrices of rank t whose differences are also of rank t. We therefore ask the following question, which is generally open:

Can any focal-spread be extended to a spread set for a translation plane?

For example, consider a focal-spread of dimension $t+1$, then looking at the $1 \times t$ spread sets, whose differences are of rank 1, it is clear that such sets can obviously be extended in many ways to t-spreads. So the question only is relevant for $k > 1$.

3 Inherited groups

9 Definition. In a focal-spread of dimension $t+k$ over $GF(q)$ and focus L of dimension t, every collineation is assumed be an element of $\Gamma L(t+k,q)$, leave invariant the focus L and permute the components of the partial Sperner k-space $\mathcal{S}$.

A 'homology' h is a collineation of $GL(t+k,q)$ with the following properties: (1) h leaves invariant the focus L and another k-subspace of $\mathcal{S}$, (2) h fixes one of the two fixed components pointwise and acts fixed-point-free on another fixed component (in the case that a k-component is fixed pointwise, we assume that the group acts fixed-point-free on the focus).

10 Remark. (1) An homology of a focal-spread fixes exactly two components and permutes the remaining components semi-regularly. The pointwise fixed subspace is called the 'axis' of h and the fixed subspace is the 'coaxis' of h.

(2) If a collineation h fixes the focus pointwise then h is an homology.

Proof. Represent the focal-spread in the form

$$x = 0,\ y = 0,\ y = xM;\ M \text{ is a } k \times t \text{ matrix in set } \mathcal{M}$$

for x a k-vector and y a t-vector, where $x = 0$ is the focus, and $y = 0$ is a k-space. Assume furthermore that $I_{k \times t} \in \mathcal{M}$. Then any collineation in $GL(t + k, q)$ that fixes a component pointwise may be represented in either the form $l_A : (x, y) \to (xA, y)$, where A is a non-singular $k \times k$ matrix or $r_B : (x, y) \to (x, yB)$, where B is a non-singular $t \times t$ matrix. First assume that r_B is an homology, so that B acts fixed-point-free on the focus. Assume that B fixes $x = 0, y = 0$, and $y = xM$. So, $xMB = xM$. Choose then any $x_0M = z_0$, for z_0 a t-vector, so that $z_0B = z_0$. However, this is contrary to the action of B. Since the previous argument is also valid for any power $B^j \neq I_t$, then $\langle r_B \rangle$ fixes exactly two components and acts semi-regularly on the remaining components. This proves (1).

Now assume that h fixes the focus pointwise and hence may be represented as l_A, where A is a non-singular $k \times k$ matrix. Assume that $\langle l_A \rangle$ is not fixed-point-free on the the k-component $y = 0$. Without loss of generality, let $x_0A = x_0$, where x_0 is a non-zero k-vector. A component $y = xM$ maps to $y = xA^{-1}M$, under l_A. Since (x_0, x_0M) is a non-zero vector common to $y = xM$ and $y = xA^{-1}M$, it follows that l_A fixes each k-component and since this means l_A is in the kernel. Since the identity mapping fixing each component is also in the kernel and the kernel is a field by Remark 3, it follows that $A - I_k$ is non-singular, a contradiction to the fact that A has 1 as an eigenvalue. Hence, $\langle l_A \rangle$ acts semi-regularly on $y = 0$, so that the group is an homology group. $\boxed{QED}$

11 Definition. An 'elation' e of a focal-spread of type (t, k) over $GF(q)$ is a collineation of $GL(t + k, q)$ with the following two properties: (1) e fixes the focus L pointwise, and (2) if V is the associated vector space of dimension $t + k$ over $GF(q)$, e fixes V/L pointwise.

12 Remark. Choose coordinates so that the focal-spread may be represented in the form:

$$x = 0,\ y = 0,\ y = xM;\ M \text{ is a } k \times t \text{ matrix in set } \mathcal{M}$$

for x a k-vector and y a t-vector.

(1) Any collineation e in $GL(t+k,q)$ that fixes the focus $x=0$ pointwise may be represented in the form $e_{A,C}:(x,y)\to(xA,xC+y)$, where A is a non-singular $k\times k$ matrix, where C is a $k\times t$ matrix of rank k.

(2) e is an elation if and only if in the representation $e_{A,C}$ then $A=I$ and where C acts additively and semi-regularly of order p (for $q=p^r$, p a prime) on $\mathcal{M}\cup\{0\}$.

Furthermore, any group E of elations is an elementary Abelian p-group that acts semi-regularly on the partial Sperner k-space.

Proof. If e is an elation the e fixes $V/(x=0)$ pointwise and it follows that $A=I_k$ and therefore C is non-zero. Since e permutes the partial Sperner k-space, we have $y=0$ mapping to $y=xC$, so C has rank k. Furthermore, $y=xM$ maps to $y=x(M+C)$, since $M+B\neq M$ then the elation group is semi-regular and clearly the order of any elation then e fixes is p. The converse is immediate and left to the reader as the last part of the part (2) follows immediately from the matrix form of an elation. $\boxed{QED}$

Finally, in this section, we briefly introduce the type of groups that can act on focal-spreads that are k-cuts and are inherited from the group of a t-spread. The following lemma was pointed out to the authors by one of the referees, which is easily verified.

13 Lemma. *Let $V=V(2t,q)$ and let S be a t-spread on V and assume that G is a collineation group of S. Let W be a subspace of dimension $t+k$ containing a component L and assume that both W and L are G invariant. Then G acts faithfully as a group of collineations of the k-cut $F=S\setminus W$.*

So, obviously, any kernel homology group of the translation plane of order q^t that leaves k-subspaces invariant inherits as a collineation group of any k-cut.

14 Corollary. *(1) Any affine homology group with axis $y=0$, and coaxis $x=0$, of the t-spread inherits as a collineation group of a k-cut focal-spread with focus $x=0$.*

(2) Any affine elation group with axis $x=0$ of the t-spread inherits as a collineation group of any k-cut focal-spread with focus $x=0$.

Proof. Consider the axis $y=0$ of an affine homology group. Take the subspace V_{t+k} generated by any k-subspace of $y=0$ and $x=0$. Then the affine homology group will leave V_{t+k} invariant. This proves (1). Let E be an elation group with axis $x=0$. An elation group acts trivially on the quotient space $V_{2t}/(x=0)$ and hence will leave V_{t+k} invariant. This proves (2). $\boxed{QED}$

We now formally define the planar extension of a focal-spread admitting a particular group. This definition was suggested by one of the referees.

15 Definition. Let F be a focal-spread of type (t,k) with collineation group G in $\Gamma L(t+k,q)$ and let V be a $2t$-dimensional vector space over $GF(q)$. Let W be a $t+k$-dimensional subspace of V containing the focus L of F. Then we shall say that F has a 'planar extension with group G' if and only if there is a t-spread S of V and a group G' in $\Gamma L(2t,q)$, such that there is a $GF(q)$-linear monomorphism $\phi : F \to V$ and a group isomorphism $f : G \to G'$ such that $\phi(L)$ lies in a unique spread component L' of S, where $\phi(L^{\alpha}) \subseteq L'^{f(\alpha)}$, for $\alpha \in G$.

When $G = \langle 1 \rangle$, we simply say that F has a 'planar extension'.

Note that planar extensions of focal-spreads need not be unique.

Now we show at least one situation where a focal-spread necessarily is a k-cut. As a guide to the argument that we give, we formulate a matrix-based method to construct k-cuts from t-spreads (corresponding to translation planes of order q^t with kernel containing $GF(q)$). The situation which we then consider will show almost immediately that the focal-spread in question can arise as a k-cut.

16 Remark. Let π be a translation plane of order q^t, and kernel containing $GF(q)$. Represent points of the $2t$-dimensional $GF(q)$-vector space as $(x_1,\ldots,x_t,y_1,\ldots,y_t)$, where x_i, $y_i \in GF(q)$, for $i = 1,2,\ldots,t$. Assume that we have a matrix t-spread set

$$x = 0, y = 0, y = xM,\ M \in \mathcal{M},$$

where $\mathcal{M}$ is a set of non-singular $t \times t$ matrices, where the differences of distinct matrices are also non-singular, and where $x = (x_1,\ldots,x_t)$, $y = (y_1,\ldots,y_t)$. Let V_{t+k} be the $t+k$-dimensional subspace of vectors $(x_1,\ldots,x_k,0,\ldots,0,y_1,y_2,\ldots$ $\ldots,y_t)$ for all $x_j, y_i \in GF(q)$, for $j = 1,2,\ldots,k$, $i = 1,2,\ldots,t$. If we form the k-cut, $V_{t+k} \cap Z$, where Z is one of the components of the matrix t-spread, we have the focal-spread:

$$\begin{aligned}
&V_{t+k} \cap (x=0) = (x=0),\\
&V_{t+k} \cap (y=0) = \{(x_1,\ldots,x_k,0,0,\ldots,0); x_i \in GF(q), i = 1,2,\ldots,k\},\\
&V_{t+k} \cap (y=xM) = \{(x_1,\ldots,x_k,0,\ldots,0),(x_1,\ldots,x_k,0,\ldots,0)M\},\ \forall M \in \mathcal{M}.
\end{aligned}$$

Now if $I_{k\times t}$ is the $k \times t$ matrix with 1 in the (i,i) positions, for $i = 1,2,\ldots,k$ and zeros in all other positions, then $(x_1,\ldots,x_k,0,\ldots 0)M = (x_1,\ldots,x_k)I_{k\times t}M$. Now suppress the $t = k$ zeros in x and now use x to represent $(x_1,\ldots,x_k)$. Then the focal spread of type (t,k) over $GF(q)$ is

$$x = 0, y = 0, y = xI_{k\times t}M, \text{ for } M \in \mathcal{M},$$

where x is a k-vector and y is a t-vector.

17 Theorem. *Let $\mathcal{F}$ be a focal-spread of type (t,k) over $GF(q)$. Assume that there is an affine group G of order q^t-1 fixing the focus, fixing a k-component pointwise and acting transitively on the remaining k-spaces of the partial Sperner k-spread.*

(1) Then G is an homology group (every non-identity element of G is an affine homology with the same axis and coaxis)

(2) There is a nearfield plane π of order q^t so that $\mathcal{F}$ is a k-cut of π. Hence, there is a planar extension with group G.

Proof. Represent the focal-spread in the form

$$x=0,\ y=0,\ y=xM;\ M \text{ is a } k\times t \text{ matrix in set } \mathcal{M}$$

for x a k-vector and y a t-vector, where $x=0$ is the focus, and $y=0$ is a k-space. Assume furthermore that $I_{k\times t}\in\mathcal{M}$. Suppose a focal-spread of type (t,k) admits a collineation group that fixes the focus and one k-component pointwise and is transitive on the remaining k-components. Choosing $x=0$ as the focus and $y=0$ as the k-component that is pointwise fixed, we have vectors (x,y), where x is a k-vector and y is a t-vector. The collineation group has elements of the form $\sigma_B:(x,y)\to(x,yB)$, where B is a non-singular $t\times t$ matrix. We may always choose a representation so that $y=xI_{k\times t}$ is a k-component. Therefore, we have $y=xI_{k\times t}B$ as a k-component for all elements τ_B. Assume that some σ_B does not act fixed-point-free on the focus $x=0$. Without loss of generality, let $y_0B=y_0$, where y_0 is a non-zero t-vector (that is, $(0,y_0)$ is fixed by σ_B). There exists a non-zero k-vector x_0 and a matrix M of $\mathcal{M}$ so that $x_0M=y_0$. Hence, $(x_0,x_0MB)=(x_0,y_0B)=(x_0,y_0)=(x_0,x_0M)$, so that (x_0,y_0) is a vector on $y=xMB$ and $y=xM$, so this means that σ_B leaves $y=xM$, invariant, a contradiction to our assumptions. Hence, G is a homology group and as such acts sharply transitive on the non-zero vectors of the focus $x=0$. In the context of Remark 16, now consider x and y t-vectors and form the associated $2t$-dimensional vector space over $GF(q)$ with vectors $(x_1,\dots,x_t,y_1,\dots,y_t)$. Let $\mathcal{C}$ denote the group $\{B;\sigma_B=\begin{bmatrix} I_k & 0\\ 0 & B\end{bmatrix}\in G\}$. Form the putative t-spread:

$$x=0, y=0, y=xB,\ B\in\mathcal{C}.$$

Now we claim that this is a t-spread. To see this, we note that if $y=xB$ and $y=xD$, for $B,D\in\mathcal{C}$, share a vector $(x_0,x_0B)=(x_0,x_0D)$, for $x_0\neq 0$, then $x_0BD^{-1}=x_0$. However, $\mathcal{C}$ is fixed-point-free, as noted above. Hence, we obtain a t-spread and by Remark 16, it follows immediately the focal-spread is a k-cut. This completes the proof of the theorem.

This completes the proof. $\boxed{QED}$

4 Additive Focal-Spreads

We now consider the focus as $x = 0$ and $y = 0$ a k-space. If the group fixing the focus is an elation group transitive on the Sperner k-space and fixes $x = 0$ pointwise, in contrast to Theorem 17, it is not known that the focal-spread is a k-cut of a semifield plane. If we assume that a group of order q^t fixes $x = 0$ pointwise, (and fixes no other points) then the group will act sharply transitive on the set of q^t k-spaces of the Sperner k-space.

18 Definition. A focal-spread of type (t, k) over $GF(q)$ shall be said to be 'additive' if there is an elation group E of order q^t with axis the focus.

19 Remark. Choose a representation for the focal-spread as in Remark 12. Then if there is an elation group E of order q^t with axis the focus, then there is a matrix spread set for the focal-spread as follows:

$$x = 0, y = xC;\ C \in \mathcal{M},$$

where $\mathcal{M}$ is an elementary Abelian p-group of order q^t of $k \times t$ matrices. The group $E = \left\langle e_C : (x, y) \to (x, xC + y) = (x, y) \begin{bmatrix} I_k & C \\ 0 & I_t \end{bmatrix}; C \in \mathcal{M} \right\rangle$.

Thus, an additive focal-spread has a matrix representation whose matrix set is an additive group of $k \times t$ matrices.

In this section, we consider a generalization of semifield spreads; additive focal-spreads. Hence, coordinate and construction methods useful for the study of semifields should be applicable in analysis of additive focal-spreads. Indeed, methods of Knuth [11] and Kantor [10] are very much in use in a more general setting.

Suppose that we have an additive focal-spread. This means we have a focal-spread of type (t, k) over $GF(q)$, for $q = p^r$, p a prime. Choose $x = 0$ as the focus and $y = 0$ as a fixed k-space. Then the focal-spread is $x = 0, y = xM$, where M is in a set S of q^t $k \times t$ matrices of rank k or the zero matrix, such that the difference of any two distinct matrices in S is also of rank k and where S is an elementary Abelian p-group and hence a vector space.

The following result now follows immediately from Remark 19.

20 Theorem. *An focal-spread of type (t, k) that admits an elementary Abelian matrix spread set, is an additive focal-spread (admits an elation group*

q^t that fixes the focus pointwise and acts transitive on the Sperner partial k-spread).

Now if we consider the vector space over $GF(p)$, for $q = p^r$, for p a prime, then we have a set of p^{tr} $kr \times tr$ matrices and each row i is clearly a function f_i of the vectors in row. So row i is $f_i(x)$, where x is a kt-vector over $GF(q)$, and if two matrices M and N in S have the same row i, then $M - S$ cannot have rank k. Now if S is additive, clearly this implies that the functions f_i are also additive. Since we are working over $GF(p)$, this means that the functions f_i are linear transformations over $GF(p)$. Write $f_i(x) = xA_i$, where x is the kt-vector in row 1 of the matrix M_x of S. Hence, we have a set of k $kr \times kr$ matrices A_i, for $i = 1, 2, \ldots, k$ and $A_1 = I$. We note that $A_i - I$ is non-singular since the matrices M in S are all of rank k. Consider the rows xA_i and xA_j, for $i \neq j$. Then we know that $xA_i = xA_j$, which implies that $A_i - A_j$ is non-singular. Note, in this context, the $k \times t$ matrices over $GF(q)$, become $kr \times tr$ matrices over $GF(p)$. We also know that the set of k vectors $A_x = \{xA_i; i = 1, 2, .., kr\}$ is linearly independent for each non-zero vector A_x. We have that in an additive spread, the rows are xA_i where x is the first row. For each x, we have a non-singular matrix, which means that the rows are linearly independent for each fixed x.

This is equivalent to having $x(\sum_{i=1}^{kr} \alpha_i A_i) = 0$, for any x non-zero tr-vector, imply that $\alpha_i = 0$, for $i = 1, 2, \ldots, kr$.

This means that the sum over $GF(p)$ of this set of matrices is non-singular. Clearly then, the set of matrices generates an additive set of matrices. So, to recreate the original spread, we take the rows to be xA_i for $i = 1, 2, .., tr$, in the spread case.

Hence, we have shown the following theorem:

21 Theorem. *Each $t+k$-additive focal-spread over $GF(q)$, for $q = p^r$, gives rise to a partial spread of kr $tr \times tr$ matrices,*

$$y = xA_i, \text{ for } i = 1, 2, \ldots, kr$$

where $A_1 = I$ and

$$x(\sum_{i=1}^{kr} \alpha_i A_i) = 0,$$

implies $\alpha_i = 0$, for $i = 1, 2, \ldots, kr$, for any x non-zero.

Conversely, any such partial spread that can be extended to some additive spread produces a $(t+k)r$-additive focal-spread over $GF(p)$, which is a kr-cut.

Hence, an additive (t', k')-focal-spread over $GF(p)$ may be extended to a semifield t'-spread if and only if the companion additive partial spread of degree $p^{k'} + 1$ and order $p^{t'}$ may be extended to a semifield spread.

Proof. If the partial spread indicated can be extended to a spread $\mathcal{S}$ over $GF(p)$, then we obtain a set of $p^{tr}-1$ non-singular $tr \times tr$ matrices B_i, $i = 1, 2, \ldots, p^{tr}-1$, such that the spread is given by $x = 0$, $y = 0$, $y = xB_i$. Now given any nonzero tr-vector v considered as a row vector, then there is exactly one matrix $B_{1(v)}$, in which v appears as row 1. Similarly, there is exactly one matrix $B_{j(v)}$ of $\mathcal{S}$ such that v appears in row j. There is a function g_i such that row i in $B_{1(w)}$ is $g_i(w)$, where w is the tr-vector considered as row 1 in $B_{1(w)}$. We consider the set of matrices B_i in the following manner: $[w^T, g_2(w)^T, \ldots, g_{p^{tr}-1}(w)^T]^T$, where $g_i(w)$ is the ith row of the matrix $B_{i(w)}$, such that w is in row 1, and C^T denotes the transpose of the matrix C. Now if we assume that the matrices B_i are additive, then if $B_{1(v+w)} = B_{1(v)} + B_{1(w)}$, this implies that the functions g_i are additive functions. This shows that we will find the set A_i, for $i = 1, 2, \ldots, kr$ within the set of matrices B_i. Indeed, since the additive spread is itself a vector space over $GF(p)$, this also implies that there is a basis of tr-vectors, within which we will find the set A_i. Hence, we have extended the linearly independent set $\{A_i, i = 1, \ldots, kr\}$ to a basis for the additive spread, which has a basis of tr-vectors over $GF(p)$. Now we note that the additive spread obtained by this method of mappings from the first row generates the same spread as the original spread. This means we could have used the kr-cut construction as well to construct the additive focal-spread. $\boxed{QED}$

22 Definition. The partial spread obtained from an additive focal-spread as in the previous theorem shall be called the 'companion partial spread'. If we actually have an additive spread, that is, a semifield spread, we call this related semifield spread the 'companion semifield spread'.

23 Remark. In a related article (see Jha and Johnson [7]), it is shown that the companion semifield spread is the spread arising from the dual of the associated semifield.

Consider any additive partial spread of order p^t, and degree p^k. By definition, an additive partial spread is an elementary Abelian p-group so is a $GF(p)$-vector space and thus has a basis $\{A_1 = I,\ A_2, \ldots, A_k\}$ of $t \times t$ matrices such that the partial spread is $\sum_{i=1}^{k} \alpha_i A_i$, where $\alpha_i \in GF(p)$. Now consider a $t+k$-vector space over $GF(p)$ with vectors $(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_t)$. Regard $y = 0 = (y_1, \ldots, y_t)$ to be in the original additive spread. Now form

$$y = x\left[\ w^T,\ (wA_2)^T,\ (wA_3)^T,\ .\ .\ .\ .\ , (wA_k)^T\ \right]^T,$$

for all t-vectors w over $GF(p)$,

where M^T denotes the transpose of the matrix M.

We claim that this is a $t+k$-dimensional additive focal-spread with focus of dimension t. To see this we note that $\sum_{i=1}^{k} \alpha_i A_i$ is a non-singular linear transformation, for α_i not all zero and so for w non-zero t-vectors then $w\sum_{i=1}^{k}\alpha_i A_i = 0$, if and only if $\alpha_i = 0$, but $w\sum_{i=1}^{k}\alpha_i A_i = \sum_{i=1}^{k}\alpha_i w A_i$, since the α_i are in $GF(p)$. Therefore, the matrices $\left[\begin{array}{ccc} w^T, & (wA_2)^T, & (wA_3)^T,\ldots\ldots,(wA_k)^T \end{array}\right]^T$ are all of rank k and therefore we have an additive focal-spread of dimension $t+k$ with focus of dimension t. Hence, we have the following result.

24 Theorem. *Additive focal-spreads of dimension $t+k$ with focus of dimension t over $GF(p)$ are equivalent to additive partial t-spreads of degree p^k.*

25 Remark. We may always regard $x = 0$ adjoined to any additive partial spread of degree p^t to produce a partial spread of degree $p^t + 1$.

In a semifield spread, it is possible to consider the associated spread in the dual vector space. The sequences of transpose and dual of Knuth [11] and Kantor [10] generalize directly. In particular, if $x = 0, y = xM$, represents the spread then the dual spread is given by $x = 0, y = xM^T$. For additive focal-spreads, we also have a representation $x = 0, y = xM$, where now the matrices are $k \times t$ instead of $t \times t$, and it does not make sense to consider a transposed additive focal-spread. However, it is possible to transpose the partial spread corresponding to an additive focal-spread. In the semifield case, the companion spread is the semifield spread arising from associated dual semifield. So, the process would be : semifield→dual→transpose→dual. For additive focal-spreads, we also could have:

$$\begin{array}{rcl} \text{additive focal-spread} & \rightarrow & \text{companion} \rightarrow \\ \text{transpose} & \rightarrow & \text{companion (an additive focal-spread).} \end{array}$$

26 Theorem. *Let $\mathcal{A}$ be an additive focal-spread of dimension $t+k$ over $GF(p)$ with focus of dimension t. Then there is a related additive focal-spread $\mathcal{A}^{ctc}$ obtained from $\mathcal{A}$ by the iterated processes of*

$$companion \rightarrow transpose \rightarrow companion.$$

If

$$\begin{array}{rcl} x & = & 0, y = x\left[\begin{array}{ccc} w^T, & (wA_2)^T, & (wA_3)^T,\ldots\ldots,(wA_k)^T \end{array}\right]^T, \\ & & \textit{for all } t\textit{-vectors } x \textit{ over } GF(p). \end{array}$$

represents $\mathcal{A}$ then $\mathcal{A}^{ctc}$ may be given by

$$\begin{array}{rcl} x & = & 0, y = x\left[\begin{array}{ccc} w^T, & (wA_2^T)^T, & (wA_3^T)^T,\ldots\ldots,(wA_k^T)^T \end{array}\right]^T, \\ & & \textit{for all } t\textit{-vectors } x \textit{ over } GF(p). \end{array}$$

PROOF. The companion partial spread of $\mathcal{A}$ is $x = 0, y = x\sum_{i=1}^{k}\alpha_i A_i$, where $\alpha_i \in GF(p)$. Transposing this partial spread produces the partial spread $x = 0, y = x\sum_{i=1}^{k}\alpha_i A_i^T$, where $\alpha_i \in GF(p)$, and forming the companion focal-spread, we have

$$\begin{aligned} x &= 0, y = x\left[\begin{array}{ccccccc} w^T, & (wA_2^T)^T, & (wA_3^T)^T, & . & . & . & ,(wA_k^T)^T \end{array}\right]^T, \\ & \quad \text{for all } t\text{-vectors } w \text{ over } GF(p). \end{aligned}$$

$\boxed{QED}$

4.1 Nuclei of Additive Focal-Spreads

It is well known that given a finite semifield spread, the middle and right nuclei and left nucleus N_M, N_R N_L are fields and, whose multiplicative groups are given by the right and left homology groups $\left\langle \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \right\rangle$, $\left\langle \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} \right\rangle$, $\left\langle \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \right\rangle$for $A \in N_M - \{0\}$ and $B \in N_R - \{0\}$, $M \in N_L - \{0\}$. Under the dualization process, the middle nucleus of a semifield is isomorphic to the middle nucleus of the dual semifield, whereas the right and left nuclei are interchanged in the process. The question for this subsection is whether the corresponding homology groups are interchanged between an additive focal-spread and the companion additive partial spread.

First assume that we have an additive focal-spread

$$\begin{aligned} x &= 0, y = x\left[\begin{array}{ccccccc} w^T, & (wA_2)^T, & (wA_3)^T, & . & . & . & ,(wA_k)^T \end{array}\right]^T, \\ & \quad \text{for all } t\text{-vectors } x \text{ over } GF(p). \end{aligned}$$

Assume also that $\begin{bmatrix} I_k & 0 \\ 0 & B \end{bmatrix}$ is a collineation of the additive focal-spread. The image of the additive focal-spread is

$$\begin{aligned} x &= 0, y = x\left[\begin{array}{ccccccc} (wB)^T & (wA_2B)^T, & (wA_3B)^T, & . & . & . & ,(wA_kB)^T \end{array}\right]^T, \\ & \quad \text{for all } t\text{-vectors } x \text{ over } GF(p). \end{aligned}$$

also represents the additive focal-spread. This implies that

$$wA_iB = wBA_i,\ 2, 3, \dots, k.$$

for all w. This implies that A_i commutes with B, for all $i = 1, 2, \dots, k$. Hence, considering the corresponding additive partial spread

$$x = 0,\ y = x\sum_{i=1}^{k}\alpha_i A_i, \text{ for all } \alpha_i \in GF(p),$$

admits the collineation $\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$ fixing all components. We note that we are 'not' assuming that B has an inverse.The converse is also valid, as is easily verified. Thus, we obtain the following proposition. If we define the nuclei corresponding to a set of particular collineation as the set of matrices that define the collineations union the zero mapping, we then have the following proposition.

27 Proposition. *The left nucleus of an additive partial spread is isomorphic to the right nucleus of the associated companion additive focal-spread.*

28 Remark. (1) It is easily verified that the right and middle nuclei of an additive partial spread are fields, whereas the left nucleus, which corresponds to group that fixes each component may not be a field.

(2) For example, consider any regulus $\mathcal{R}$ of order p^t and degree $1 + p^k$ in standard form

$$x = 0, y = x\alpha; \alpha \in GF(p^k), \text{ where } k = t/2.$$

α is a scalar matrix αI_t, which then commutes with any $t \times t$ matrix M. Assuming that we non-singular matrices, then the left nucleus contains $GL(2, p^k)$.

(3) Hence, we cannot normally determine that the right nucleus of an additive focal-spread is a field.

We note in Proposition 30 that the middle nucleus of an additive partial spread is a field, whereas the right nucleus (if $k/2 \leq t$) is a ring but may not be a field.

29 Remark. In the following, for simplicity, we take additive focal-spreads of type (t, k) over the prime field $GF(p)$.

The following proposition was suggested by one of the referees of the original article.

30 Proposition. *Let $\mathcal{F}$ denote a matrix set of $k \times t$ matrices over $GF(p)$, containing the zero matrix and whose other matrices are of rank k, and whose distinct differences are also of rank k.*

(1) Let $\boldsymbol{R}(\mathcal{F}\,) = \{A;\ A$ is a $k \times k$ $GF(p)$ matrix such that $A\mathcal{F} = \mathcal{F}\}$. Then $\boldsymbol{R}(\mathcal{F}) \cup \{0_{k\times k}\}$ is a field.

31 Definition. (2) Let $\boldsymbol{L}(\mathcal{F}\,) = \{B;\ B$ is a $t \times t$ $GF(p)$ matrix such that $\mathcal{F} = \mathcal{F}A\}$. Then $\boldsymbol{L}(\mathcal{F}) \cup \{0_{k\times k}\}$ is a ring, which is a field if $k > t/2$.

Proof. If B, $C \in \boldsymbol{L}(\mathcal{F})$, assume rank of $B + C = D < t$. Assume that $y_0 D = 0$, for y_0 a non-zero t-vector. For any x_0 non-zero k-vector, there is a $k \times t$ matrix M of $\mathcal{F}$ so that $x_0 M = y_0$. Therefore, $(x_0, x_0 MD) = (x_0, 0)$. It follows that $MD = 0_{k\times t}$. Row-reduce to note that this implies that $I_{k\times t}D = 0_{k\times t} = D_1$, where D_1 is the submatrix of D of the first k rows of D. Hence, the kernel of D is at least k-dimensional. Now since $NB + NC$ is a k-component for all k-components N, it follows that D is of rank at least k. Therefore, the rank$D$ and kernel$D \geq 2k$. Hence, if $k > t/2$, this is a contradiction. $\boxed{QED}$

5 Associated Designs, Double and Triple-Spreads

32 Definition. A 'double-spread' of a vector space is a partition using subspaces of two distinct dimensions. So a focal-spread is also a double-spread. Similarly, a 'triple-spread' is a partition of a vector space into subspaces of three distinct dimensions.

Using the idea of a general focal-spread, in this section, we find examples of both double-spreads, which are not focal-spreads, and triple-spreads.

33 Lemma. *Suppose that B is a focal-spread of type $(1+k, k)$ over $GF(q)$. Then each hyperplane that intersects the focus in a k-dimensional subspace induces a partition of a vector space of dimension $2k$ over $GF(q)$ by $q+1$ subspaces of dimension k and $q^{k+1} - q$ subspaces of dimension $k-1$. Hence, each hyperplane then produces a double-spread.*

Proof. For a focal-spread of type $(1 + k, k)$, with focus L, consider any hyperplane H, a subspace of dimension $2k$, that intersects L in a subspace of dimension k. Since $2k + k - (2k + 1) = k - 1$, then H intersects each k-component in at least a $k-1$-subspace. Let a denote the number of k-dimensional intersections so that $q^{k+1} - a$ is the number of $k-1$-dimensional intersections.

$$a(q^k - 1) + (q^{k+1} - a)(q^{k-1} - 1) = q^{2k} - q^k,$$

which implies that

$$aq^{k-1}(q^k - 1) = q^k(q^k - 1),$$

so that $a = q$. $\boxed{QED}$

In a similar manner, triple-spreads may be constructed. The following may be proved in a manner similar to the previous lemma and will be left to the reader.

34 Remark. Given any focal-spread of type (t, k) with focus L. Then any subspace H_{2k-1} of dimension $2k-1$ that intersects L in a k-dimensional subspace is partitioned by a subspace of dimension k, q^2 subspaces of dimension $k-1$, and $(q^{k+1} - q^2)$ subspaces of dimension $k-2$. Thus, in this way, a triple-spread is obtained.

Finally, we construct associated designs.

35 Theorem. *Let $\mathcal{F}$ be any focal-spread of type $(1+k,k)$ with focus L. Denote by $\boldsymbol{D} = \boldsymbol{D}(\mathcal{F}) = (\boldsymbol{P}, \boldsymbol{B})$ the incidence structure whose point set $\boldsymbol{P} = \mathcal{F} - \{L\}$ and whose blocks $\boldsymbol{B}$ are hyperplanes that do not contain L. Then D is an affine (or resolvable) $2-(q^{k+1}, q, 1)$-design. Any parallel class is formed from the hyperplanes that intersect L in a common k-space.*

Proof. By Lemma 33, every block contains q points.

Since $|\boldsymbol{B}| = (q^{2k+1}-1)/(q-1) - (q^k-1)/(q-1) = q^k(q^{k+1}-1)/(q-1)$, there are exactly q^k hyperplanes that intersect L in a common k-space U. Since any point P lies in the hyperplane $P+U$, it follows that this set of hyperplanes covers the points. Furthermore, two hyperplanes containing U cannot share a point. In addition, the points P and Q lines in a unique hyperplane $P+Q$ of $\boldsymbol{B}$. This completes the proof. $\boxed{QED}$

6 Additive Focal-Spreads of Large and Small Degrees

In this section, we consider additive focal-spreads of type $(t,2)$ and $(t,t-1)$. The focal-spreads of type $(t,2)$-spreads over $GF(p)$ can always be extended and the focal-spreads of type $(t,t-1)$ can either be extended or produce interesting additive maximal partial spreads.

An additive focal-spread of type $(t,2)$ over $GF(p)$, for p a prime, is a set of p^t $2\times t$-matrices over $GF(p)$. Then, we have a partial spread $x=0, y=x, y=xA_1$, where A_1 is a non-singular $t\times t$ matrix over the prime field.

Change coordinates by $(x,y)\to(x,-xA_1+y)$ and then by $(x,y)\to(x, -yA_1^{-1})$, to convert to $x=0, y=x, y=0$, which clearly may be extended to an additive spread. By changing coordinates back, we have extended the original partial spread to an additive spread. This proves the following:

36 Theorem. *(1) Any additive focal-spread of type $(t,2)$ over $GF(p)$ may be extended to a semifield spread.*

(2) Given any additive focal-spread of type $(t,2)$ over $GF(q)$, let g denote the function that maps the t-vectors of row 1 to the t-vectors of row 2 in the associated $2\times t$ additive matrix set. If g is a linear function over $GF(q)$, then the additive focal-spread of type $(t,2)$ may be extended to a semifield spread.

We now turn to the focal-spreads of type $(t,t-1)$.

37 Theorem. *For any additive focal-spread of type $(t,t-1)$ over $GF(p)$, if the associated companion additive partial spread is not maximal then the additive focal-spread can be extended to a semifield spread.*

Therefore, every additive focal-spread of type $(t, t-1)$ over $GF(p)$ gives rise to a semifield plane or an additive maximal partial spread.

PROOF. Assume that we have an additive $(t, t-1)$–focal spread over $GF(p)$. Then we obtain a set of $t-1$ $t \times t$ matrices I, $A_2, \ldots, A_{t-1}$ such that

$$x(\sum_{i=1}^{t-1} \alpha_i A_i) = 0,$$

for all $\alpha_i, i = 1, 2, .., t-1$ in $GF(p)$, then either all $\alpha_i = 0$ or $x = 0$. So we have an additive partial t-spread of degree $1+$ p^{t-1}. Assume that the partial spread $1 + p^{t-1}$ t-subspaces is not maximal so let $y = xM$ extend the set $\{y = xA_i$, $y = x$ and $x = 0\}$. Therefore,

$$(*) : x(\sum_{i=1}^{t-1} \alpha_i A_i) = xM,$$

for α_i, $i = 1, 2, \ldots, t-1$ in $GF(p)$ if and only if $x = 0$. So consider

$$x(\sum_{i=1}^{t-1} \alpha_i A_i + \beta M) = 0$$

for any set α_i, $i = 1, 2, .., t-1, \beta \in GF(q)$, not all of which are zero and we may assume that β is not zero and some α_j is not zero. Then

$$x(\sum_{i=1}^{t-1} \frac{\alpha_i}{-\beta} A_i + M) = 0$$

which implies by $(*)$ that $x = 0$. Therefore, the set of p^t matrices

$$\sum_{i=1}^{t-1} \alpha_i A_i + \beta M$$

forms an additive spread set. Hence, either the partial spread is maximal or there is a semifield plane. Let $M = A_t$. We know that considering the first row of the focal-spread as x then xA_i for $i = 1, 2, \ldots, t-1$ gives the i *th* row of the $(t-1) \times t$ matrix. Therefore, xA_t gives a t th row that extends the focal-spread to a spread. This completes the proof of the theorem. $\boxed{QED}$

7 Reconstruction

We begin with an open problem.

38 Problem. Does every $2-(q^{k+1},q,1)$-design with the assumptions on parallel classes given previously arise from a focal-spread?

In this section, we show that if the design can be embedded into a vector space of dimension $2k+1$, then with certain assumptions on the point and line set, we reconstruct a focal spread.

We recall that in a $2-(q^{k+1},q,1)$-design, there are $q^k(q^{k+1}-1)/(q-1)$ blocks, of which there are exactly $(q^{k+1}-1)/(q-1)$ containing a given point. If the design can be embedded into a vector space of dimension $2k+1$ over $GF(q)$, where the blocks are hyperplanes, there are exactly $(q^k-1)/(q-1)$ hyperplanes not in the block set. Hence, we prove the following 'reconstruction theorem'.

39 Theorem. *Let $\boldsymbol{D}=(\boldsymbol{P},\boldsymbol{B})$ be a $2-(q^{k+1},q,1)$-design with the following properties:*

(a) The points are subspaces of a $(2k+1)$-dimensional $GF(q)$-space V.

(b) The blocks are hyperplanes of V.

Then $\boldsymbol{D}=\boldsymbol{D}(\mathcal{F})$, where $\mathcal{F}$ is a focal-spread of type $(k+1,k)$.

Proof. (1) Every point space P has dimension k and all hyperplanes containing P are in $\boldsymbol{B}$: Given a point P, there are $(q^{k+1}-1)/(q-1)$ hyperplanes of $\mathcal{H}$ containing P. If P is an s-space, then there are $(q^{2k+1-s}-1)/(q-1)$ hyperplanes containing P. Let $\mathcal{H}$ denote the set of $q^k(q^{k+1}-1)/(q-1)$ hyperplanes of the design. Since there are exactly $(q^{2k+1}-1)/(q-1)$ hyper-planes, there are exactly $(q^k-1)/(q-1)$ hyperplanes that do not belong to $\mathcal{H}$. There are exactly $(q^{k+1}-1)/(q-1)$ hyperplanes of $\mathcal{H}$ that contain P. This means that the maximum number of hyperplanes that contain P is $(q^{k+1}-1)/(q-1)+(q^k-1)/(q+1)$. Hence,

$$(q^{k+1}-1)/(q-1)+(q^k-1)/(q-1)\geq(q^{2k+1-s}-1)/(q-1),$$

or equivalently,

$$q^{k+1}+q^k\geq q^{2k+1-s}+1>q^{2k+1-s}$$

Hence, $q^{k+1}+q^k>q^{2k+1-s}$ so that $q>q^{k+1-s}-1$. Therefore, $q\geq q^{k+1-s}$, which implies that $1\geq k+1-s$ or rather that $s\geq k$. But, $s\not>k$ since otherwise there could not be $(q^{k+1}-1)/(q-1)$ hyperplanes containing P. Therefore, every point space is k-dimensional and so every hyperplane containing P lies in $\boldsymbol{B}$.

(2) $\boldsymbol{P}$ is a partial spread of k-spaces on V:

Assume that the subspace $\langle P,Q\rangle$ of dimension $s \leq 2k-1$ are in exactly $(q^{2k+1-s}-1)/(q-1)$ hyperplanes. So, there are at least $(q^2-1)/(q-1)$ hyperplanes containing P and Q and these are the hyperplanes of $\boldsymbol{B}$ contains P. But, there is exactly one block containing P and Q. Therefore, the dimension of $\langle P,Q\rangle$ is $2k$.

(3) Let $\mathcal{N}$ denote the set of hyper-planes that do not belong to $\boldsymbol{B}$. Then $L = \cap_{H\in\mathcal{N}} H$ is a $(k+1)$-dimensional subspace that interests trivially with all points.

Set $\Gamma = (V-\{0\}) - \cup_{P\in\boldsymbol{P}}(P-\{0\})$. Then $|\Gamma| = q^{k+1}-1$ by (1). Moreover, $H \in \mathcal{N}$ does not contain any point by (1), since $\dim H\cap P = k-1$ for $P \in \boldsymbol{P}$. This shows

$$|(H-\{0\}) - \cup_{P\in\boldsymbol{P}}((P\cap H)-\{0\}| = q^{k+1}-1,$$

that is, it follows that $\Gamma \subseteq H$ and $\Gamma \subseteq L$. Therefore, the $\dim L = \ell \geq k+1$. As L is contained in exactly $(q^{2k+1-\ell}-1)/(q-1)$ hyperplanes but there are exactly $(q^k-1)/(q-1)$ hyperplanes of V that are not in $\mathcal{H}$. Therefore, $\dim L = k+1$, which completes the proof of (3).

By (1), (2), (3), $\mathcal{F} =\boldsymbol{P} \cup \{\boldsymbol{L}\}$ is a focal-spread of type $(k+1,k)$ and $\boldsymbol{B}$ is the set of hyperplanes that do not contain L. This completes the proof of the theorem. $\boxed{QED}$

40 Corollary. *Assume that a* $2-(q^{k+1},q,1)$*-design can be embedded into a vector space of dimension* $2k+1$ *such that the points are vector subspaces and the lines are hyperplanes. Then there is a resolution of the lines into* $(q^{k+1}-1)/(q-1)$ *parallel classes of* q^{k+1} *lines each.*

We now turn to the question of isomorphism of the $2-(q^{k+1},q,1)$-designs that may be constructed from focal-spreads of type $(k+1,k)$. It is possible that two designs may be isomorphic by a mapping that is not in the associated group $\Gamma L(2k+1,q)$, but here we consider any isomorphism to arise in this way. In this section, we show that two focal-spreads of type (t,k) over $GF(q)$ are isomorphic by an element of $\Gamma L(2k+1,q)$ if and only if the two corresponding $2-(q^{k+1},q,1)$ designs are isomorphic by an element of $\Gamma L(2k+1,q)$.

41 Definition. Suppose two focal-spreads F_1 and F_2 of dimension $2k+1$ over $GF(q)$, with foci of dimension $k+1$ are isomorphic. We define an 'isomorphism' as a element of $\Gamma L(2k+1,q)$ that maps one Sperner k-spread to the second Sperner k-spread and hence maps the focus of one focal-spread to the focus of the remaining focal-spread. Clearly, we may identify the two foci.

Similarly, we define an isomorphism between the $2-(q^{k+1},q,1)$-designs constructed from two focal-spreads to be an element of $\Gamma L(2k+1,q)$.

42 Theorem. *Let* F_1 *and* F_2 *be two focal-spreads of dimension* $2k+1$ *over* $GF(q)$ *with foci of dimension* $k+1$*. Let* D_1 *and* D_2 *denote the* $2-(q^{k+1},q,1)$*-*

designs constructed from F_1 and F_2, respectively. Then F_1 is isomorphic to F_2 by an element of $\Gamma L(2k+1,q)$ if and only if S_2 is isomorphic to S_1 by an element of $\Gamma L(2k+1,q)$.

Proof. Clearly, the stabilizer of a k-subspace X of the focus L will permute the hyperplanes intersecting L in X and hence will permute the parallel class $\mathcal{P}_X$ of hyperplanes containing X. Furthermore, the stabilizer of a hyperplane containing X must leave invariant the Sperner subspread S_X of q k-subspaces of intersection. In general, take any two disjoint k-subspaces P and Q of the Sperner k-spread S_1 of F_1 and let σ be an isomorphism from F_1 onto F_2, for σ an element of $\Gamma L(2k+1,q)$ that leaves the common focus L invariant. Then σ maps S_1 to S_2, the associated Sperner k-partial spreads of F_1 and F_2, respectively. Then $P\sigma$ and $Q\sigma$ are disjoint k-subspaces of S_2 and since $\langle P,Q\rangle$ is then mapped to $\langle P,Q\rangle\,\sigma$, it follows that $\langle P,Q\rangle\cap S_1$ is mapped to $\langle P,Q\rangle\,\sigma\cap S_2$. This means that an isomorphism of focal-spreads induces an isomorphism on the associated designs D_1 and D_2, since the 'points' are the elements of D_i and the 'lines' are the hyperplanes of V_{2k+1} that intersect the common focus L in k-subspaces. Now assume that τ is an isomorphism from D_1 onto D_2. We again regard τ as an element of $\Gamma L(2k+1,q)$ that maps points of D_1 to points of D_2 and so maps S_1 onto S_2. The line set is the set of hyperplanes that intersect a fixed $k+1$-space in k-subspaces. Hence, we again may identity the fixed $k+1$-space L in each design. By requirement, τ will map a parallel class of D_1 onto a parallel class of D_2 and map the set of hyperplanes that intersect L in k-spaces back into itself. Every such hyperplane is of the form $\langle P,Q\rangle$, where P and Q are elements of F_1 and τ maps this set to the hyperplane $\langle P,Q\rangle\,\tau$. Note that the intersection of a given hyperplane with the corresponding Sperner space picks out the q 'points' of each hyperplane (line). Now since we assume that τ is in $\Gamma L(2k+1,q)$, we know that k-subspaces map to k-subspaces. So, τ maps sets of q components of F_1 to sets of q components of F_2. Therefore, if P is a component, it is an element of a $S_{1,X}$ set and the image of an $S_{1,X}$ set is a $S_{2,X'}$ set, so that $P\tau$ is a component of S_2, which means that τ sets up an isomorphism between F_1 and F_2. This proves the theorem. $\boxed{QED}$

A simple case for the designs arises from translation planes of order q^2 with spreads in $PG(3,q)$, so in this case, we have a 4-dimensional vector space and $k=1$. Then from such a planar-spread, we have a $2-(q^2,q,1)$-design with $(q^2-1)/(q-1)$ parallel classes and $q(q+1)$ lines. The points are 1-dimensional vector spaces and the lines are 2-dimensional vector spaces that intersect a given 2-dimensional subspace L in 1-dimensional subspaces.

One of the referees pointed the following remark. We shall formulate a different version following this.

43 Remark. Let $\mathcal{F}$ be a focal-spread of type $(2,1)$ over $GF(q)$. Then $\boldsymbol{D}(\mathcal{F})$

$\simeq AG(2,q)$: In this case L is a hyperplane and the points of $\boldsymbol{D}$ are the points of $PG(V)$ that do not lie on the line L. Thus, $\boldsymbol{D}$ is simply the affine plane that is obtained from $PG(V)$ by removing L together with the points incidence with it.

The matrix version of this remark is as follows:

44 Proposition. *Any focal-spread of one* 2*-space and a partial Sperner* 1*-spread may be lifted to any* 2*-spread for a translation plane of order* q^2 *with spread in* $PGL(3,q)$*. The associated* $2-(q^2,q,1)$*-design is a Desarguesian affine plane.*

PROOF. The focal-spread may be represented in the form

$$x = 0,\ y = 0,\ y = xM;\ M \text{ is a } 1 \times 2 \text{ matrix},$$

where $(x,y) = (x_1, y_1, y_2)$ and M has rank 1 and the difference of any two such matrices has rank 1. There are $q^2 - 1$ matrices $[t,u]$, for r and s in $GF(q)$ so r and s take on all values independently except for $r = s = 0$. Any spread in $PG(3,q)$ corresponds to a translation plane of order q^2 and may be written in the form

$$x = 0,\ y = 0,\ y = x\begin{bmatrix} g(t,u) & f(t,u) \\ t & u \end{bmatrix};$$
$$\forall\, t,u \in GF(q), \text{ for } (t,u) \neq (0,0),$$

where g, f are functions from $GF(q) \times GF(q)$ to $GF(q)$.

Hence, we see that any focal-spread of one 2-space and a partial Sperner 1-spread of degree q^2 can be lifted to any translation plane with spread in $PG(3,q)$. It is not difficult to show that the associated design in a Desarguesian affine plane as it may be constructed from a focal-spread arising from a Desarguesian spread in $PG(3,q)$ as a $k = 1$-cut. The details are left to the reader. $\boxed{QED}$

Assume that $t > k$ and let π be a translation plane of order q^t, let V_{2t} denote the associated vector space and let S denote the associated spread. Choose any of the $q^t + 1$ components L. Choose any vector subspace of dimension $t + k$ containing L. Consider V_{2t}/L, a $2t - t = t$-vector space. Choose any subspace of dimension $t+k$ containing L. Hence, we need to choose a $2k+1-k = k+1$-space in V_{2t}/L (if $t = k+1$, then $2k+2-(k+1) = k+1$). There are

$$(q^t-1)(q^{t-1}-1)\cdots(q^{t-k+1}-1)/\left((q^k-1)(q^{k-1}-1)\cdots(q-1)\right)$$

such subspaces. We now count the number of k-cuts.

45 Remark. (1) Given a spread S of order q^t and kernel containing $GF(q)$. Then there are

$$(q^t+1)\left((q^t-1)(q^{t-1}-1)\cdots(q^{t-k+1}-1)/\left((q^k-1)(q^{k-1}-1)\cdots(q-1)\right)\right)$$

focal-spreads of dimension $t+k$ with focus of dimension t constructed using k-cuts.

(2) When $t=k+1$, there are

$$(q^t+1)(q^t-1)/(q-1)=(q^{2t}-1)/(q-1)=(q^{2(k+1)}-1)/(q-1)$$

constructed focal-spreads of dimension $2k+1$ with focus of dimension $k+1$ and this same number of $2-(q^{k+1},q,1)$-designs constructed from the focal-spreads.

8 Notes

There are a number of new directions that have developed from the study of focal-spreads. In the second part, we summarize all of these and discuss various open problems. The reader is directed to the final comments of the article [4] for this material.

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